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An Algorithm-Independent Definition of Damage Spreading—Application to Directed Percolation

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We present a general definition of damage spreading in a pair of models. Using this general framework, one can define damage spreading in an objective manner that does not depend on the particular dynamic procedure that is being used. The formalism can be used for any spin-model or cellular automaton, with sequential or parallel update rules. At this point we present its application to the Domany–Kinzel cellular automaton in one dimension, this being the simplest model in which damage spreading has been found and studied extensively. We show that the active phase of this model consists of three subphases characterized by different damage-spreading properties.

KEY WORDS: Damage spreading; directed percolation.

1. INTRODUCTION

The concept of *damage spreading* was introduced in the context of biologically motivated dynamical systems by Stuart Kauffman.[1] The question posed is whether the phase-space trajectories of two slightly different copies of a dynamic system, subjected to the same thermal noise, will stay close (or even merge) at long times or, alternatively, will they diverge?[4] Damage spreading (DS) made its first appearance in the physics literature in the mid eighties,[2,3,4] and attracted considerable interest and attention. The main reason behind this initial enthusiasm was the hope that damage

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[4] By "slightly different copies" we mean two systems that differ initially only on a small number of sites. If we find, with non-vanishing probability, that the difference between these two systems did not go to zero at long times, we say that "damage spreads."

may spread (indicating chaotic behavior) in some regions of a system's parameter space and disappear or *heal* elsewhere. This possibility intrigued researchers, since if indeed realized, it would have indicated the existence of different *dynamic phases* in various complex systems (such as spin-glasses).[4] The initial enthusiasm concerning damage spreading has abated during subsequent years; the main reason being an apparent lack of an *objective, observer-independent measure* of whether damage does or does not spread in a given system. Even for relatively simple models, such as the two dimensional ferromagnetic Ising model, different results were obtained when heat bath or Metropolis dynamics were used.[5,6] Both these dynamic procedures are phenomenological (since they satisfy detailed balance, they can be used to generate equilibrium ensembles) and the two are equally legitimate to mimic the temporal evolution of a system in contact with a thermal reservoir. If spreading or healing of damage were to indicate some intrinsic property of the system, one would not expect the result to depend on the details of exactly which phenomenological procedure was used to generate its dynamics.

The purpose of this communication is to pose the "right" question; i.e. one which has a well defined objective answer. The essence of the argument is to consider the entire family of dynamic procedures that are consistent with the physically dictated constraints of the problem. For any particular system one of *three* possibilities may hold:

1. Damage is spreading for every member of the family of dynamic procedures
2. Damage heals for every member of this family
3. Damage spreads for a subset of the possible dynamic procedures, and heals for the complementing subset.

Hence the only question regarding damage spreading that has an unambiguous, observer-independent answer is: to which of these three classes a particular system belongs?

To demonstrate the general concept introduced here we studied the simplest dynamic model in which damage spreading has been observed, the one-dimensional Domany–Kinzel (DK) cellular automaton,[8] for which we found the phase diagram presented in Fig. 1. Note that for technical reasons discussed below we postponed discussion of DS in the Ising model to a future publication.[7]

The DK automaton is a two-parameter model whose temporal evolution contains, as special cases, the bond and site directed percolation problems. The main point made by DK was *universality*: namely, that the entire family of observed transitions of the one-dimensional cellular automaton is in the directed percolation universality class. (except a special

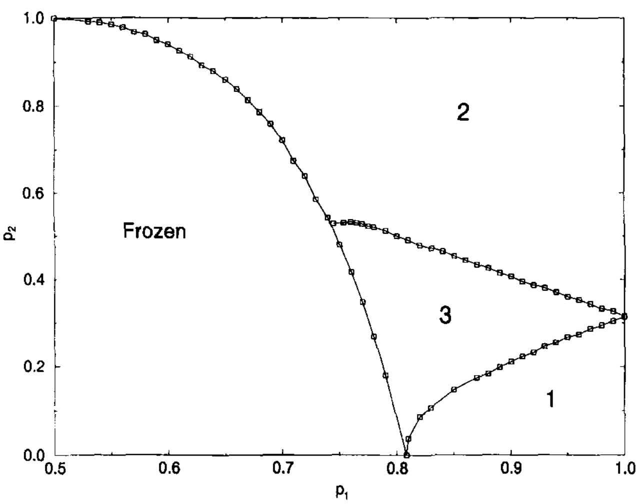

Fig. 1. Phase diagram of the Domany–Kinzel automaton. The active, percolating phase consists of three subphases; each is numbered according to the damage spreading class to which it belongs (see text).

point, for which an exact solution was presented). DK identified two phases; a "dry" or "frozen" phase, in which all initial conditions evolve to the absorbing state, and an "active" or percolating phase. Some years later Martins *et al.*[9] discovered that in a certain region of the active phase damage spreads, and it heals elsewhere. More detailed investigations, using simulations[10–13] as well as analytic (mean field) approximations[12, 14–16] confirmed the existence of this "chaotic phase." Its boundary, however, was shown[14, 15] to depend on the manner in which the dynamic procedure of the underlying DK model is carried out, while the evolution of a single replica is completely insensitive to the dynamic procedure. This prompted Grassberger[13] to observe that "it is misleading to speak of different phases in the DK automaton...instead these are different phases for very specific algorithms for simulating *pairs* of such automata." This observation is the precise analog of the problematic nature of viewing DS as a manifestation of a dynamic transition in spin models, where, as mentioned above, it was well known that different dynamics that yield identical equilibrium properties can give rise to different results for damage spreading. Thus, again, DS becomes a "subjective" concept, which is devoid of well defined meaning for the DK model, whose phases should be determined by the properties of a *single* evolving system.

The main purpose of this paper is to point out that if one defines the most general family of dynamic rules that are consistent with the physics of the problem being studied (Sec. II), DS has an objective, observer-independent meaning. Past work on DS in the DK model is reviewed in Sec. III and in Sec. IV the existence of the three well defined distinct phases described in the Introduction is established for the DK model by numerical simulations and analytical arguments.

We also tested and confirmed a recent conjecture of Grassberger, to the effect that the damage spreading transition is in the directed percolation universality class.[13] Analytical support for this conjecture came so far from approximate mean-field arguments[15] and an exact statement first made by Kohring and Schreckenberg,[14] who noted that on the $p_2 = 0$ line the dynamics of damage spreading in the DK automaton is precisely identical to the evolution of the DK automaton itself, and hence on this line DS is trivially in the DP universality class. This being a rather special line, it is of interest to try to establish such precise mapping of DS to DP also elsewhere in the $p_1 - p_2$ plane. In Sec. IV A we present such an extension.

2. RULES FOR LEGITIMATE DAMAGE SPREADING PROCEDURES

We turn now to present our arguments for the possibility of defining an observer-independent measurement of damage spreading. By this we do not mean that DS is reflected in the dynamic behavior of a single system, so that Grassberger's observation still holds; DS is a property of a *pair* of automata.[5] It is possible, however, to address the lack of objectivity implicit in one's freedom to choose the precise algorithm that is used for the evolution of the pair of replicas. If every observer can pick his favorite dynamic rule, get results (on DS) that depend on the rule used, while no measurement done on an evolving single system can differentiate between the rules—indeed it appears contradictory to claim that DS reflects "phases" of the model that is being investigated. Nevertheless such phases can be defined in a precise way.

To overcome this apparent paradox we formulate quite general and physically motivated restrictions on the possible dynamic rules that one can use for studying DS. By "physical" we mean that the restrictions are dictated by the dynamics of the single evolving system. The restrictions are as follows:

[5] In this sense DS, as defined for stochastic dynamics, differs from dynamics in deterministic nonlinear systems. For deterministic nonlinear systems one *can* find signatures of chaotic behavior in following the phase-space trajectory of a *single* evolving system. Divergence of two initially neighboring trajectories (indicating the existence of a positive Lyapunov exponent) is a computationally feasible tool to ascertain the chaotic nature of a single system's trajectory, but it is not essential to consider two replicas in order to *define* chaos.

1. The dynamic rules for the evolution of the pair of replicas are such that the evolution of a single replica is according to its "natural" dynamics.
2. The transition probability matrix for a site i for the pair of replicas can depend only on those sites that affect the evolution of site i under the dynamic rules of a single system.
3. The rules that govern evolution for the pair do not break any of the symmetries of the single-replica dynamics.

The first restriction simply means that the fact that we are watching two systems evolving in parallel should not affect the behavior of any one of them. The second constraint means that if the evolution of site i is affected, say, only by the states of its nearest neighbors, the relative states taken on site i by the two replicas should not feel longer range interactions. For example, if site i and all its neighbors j are in the same state in the two replicas, we do not expect damage to be generated at i by a damaged site which is far away (i.e. not one of the neighbors of i). The third rule implies, for example, that if there is a left-right symmetry in the evolution of a single system the same must hold for the pair of replicas.

Clearly, the subjectivity in defining the damage spreading procedure that was described above has now been shifted to this point—to selecting the restrictions that define which DS procedure is "legitimate." We do believe that there is much less arbitrariness, however, in this kind of subjectivity than what was done before, choosing, at random, one out of a continuum of physically equivalent procedures.

3. DAMAGE SPREADING IN THE DK MODEL: A BRIEF REVIEW

In this section we review briefly past work on damage spreading in the DK automaton. We emphasize the manner in which DS was calculated by various authors, and the manner in which different ways of defining DS can be embedded in a general framework.

The DK automaton is defined as follows: a binary variable $\sigma_i(t) = 0, 1$ characterizes the state of site i at (discrete) time t. $\sigma = 1$ means that the site is wet or active, whereas $\sigma = 0$ means that it is dry. The automaton evolves by a stochastic parallel update rule: the probability to obtain $\sigma_i(t+1) = 1$ for each binary variable is given by

$$P(\sigma_i(t+1) = 1 \mid \sigma_{i-1}(t), \sigma_i(t), \sigma_{i+1}(t+1)) = \begin{cases} 0 & \text{if } \sigma_{i-1} = \sigma_{i+1} = 0 \\ p_1 & \text{if } \sigma_{i-1} \neq \sigma_{i+1} \\ p_2 & \text{if } \sigma_{i-1} = \sigma_{i+1} = 1 \end{cases} \qquad (1)$$

That is, the state of site i at time $t+1$ depends only on the states of its two neighbors at time t; only wet sites can give rise to a wet site, with probabilities p_1 if one neighbor was wet and p_2 if both were wet. This model has a dry phase and a wet phase, separated by a transition line which has been determined with high accuracy by various numerical methods. In spite of its simplicity, the model has not been solved exactly, except for the special line $p_2 = 1$.[8, 17, 18] At all points on the phase boundary, except the special line, the transition to the active or wet phase is characterized by directed percolation (DP) exponents.

In order to describe DS in a more general framework, let us denote by $r_{\sigma_{i-1}, \sigma_{i+1}} = 0, 1$ the binary random variable whose value is assigned to $\sigma_i(t+1)$:

$$\sigma_i(t+1) = r_{\sigma_{i-1}(t), \sigma_{i+1}(t)} \tag{2}$$

Using this notation, the conditional probability to get $\sigma_i(t+1) = 1$ can be expressed as one-point expectation value of the random variable

$$P(\sigma_i(t+1) = 1 \mid \sigma_{i-1}(t), \sigma_i(t), \sigma_{i+1}(t)) = \langle r_{\sigma_{i-1}(t), \sigma_{i+1}(t)} \rangle \tag{3}$$

where $\langle \cdots \rangle$ denotes the average over many independent realizations. Since $r_{00} = 0$, the transition probabilities in the DK model are completely defined by the one-point expectation values of *three* random binary variables[15]

$$\langle r_{01} \rangle = \langle r_{10} \rangle = p_1, \qquad \langle r_{11} \rangle = p_2 \tag{4}$$

The one-point expectation values of these random variables specify the evolution of a *single* system. Correlations between different random variables do not affect the behavior of a single system since for each update only one of the three variables is used; if $\sigma_{i-1}(t) = \sigma_{i+1}(t) = 1$, one uses r_{11}, etc. To study damage spreading, however, we run two replicas of the system in parallel, using the same random numbers at all sites i and times t. That is, for each i, t we do generate all three numbers r_{10}, r_{01}, r_{11}, but use, on each replica, the appropriate one. For example, if the two parents of a site are $(1, 0)$ on replica A and $(1, 1)$ on B, we use r_{10} for A and r_{11} for B. Hence the temporal evolution of the entire system (and therewith damage spreading) does depend on correlations between the random variables. More precisely, the temporal evolution of n replicas is completely specified by the set of all m-point correlation functions between the random variables with $m \leqslant n$. In case of the DK model, taking the left-right symmetry into account, there are three independent correlations between the random variables:

$$\begin{aligned} \langle r_{01} r_{11} \rangle = \langle r_{10} r_{11} \rangle &= \tilde{\alpha} \\ \langle r_{01} r_{10} \rangle &= \tilde{\beta} \\ \langle r_{01} r_{10} r_{11} \rangle &= \tilde{\gamma} \end{aligned} \tag{5}$$

Since damage spreading between *two* replicas is controlled by one- and two-point functions, the parameter $\tilde{\gamma}$ will only affect the way in which *three* replicas evolve in parallel.

At this point we can explain why is the DK model so much more convenient for demonstrating our point than the one-dimensional Ising model. The DK model is defined in terms of its dynamic rules alone; since the fate of site i depends only on the states of its two neighbors, adherence to the rules 1–3 of Sec. II dictates that the random numbers may depend only on these two sites; hence the relevant parameter space of the DS problem is two dimensional and, hence, relatively easy to explore. On the other hand, the Ising model is defined by its Hamiltonian; any dynamics that satisfy detailed balance with respect to this Hamiltonian is equally acceptable. In particular, when updating spin i, one can allow its own state to affect the dynamics (as is done in the Glauber algorithm). Hence in order to include all possible algorithms that satisfy the rules for the $1-d$ Ising model, we have to allow different random numbers for each of the states of *three* spins: the two neighbors and the updated spin itself. The resulting general DS procedure depends on 14 parameters;[7] exploration of this space would have obscured the simple point we are making.

Martins *et al.*[9] were the first to address the issue of damage spreading in the DK model. Two nearly identical initial configurations were allowed to evolve on two replicas, using the same random numbers for both (the precise meaning of this statement will be explained below). They discovered that the active phase contains in fact two regions; one in which damage spreads and its complement, where it does not. The boundary between these regions was subsequently determined with increasing accuracy by Zebende and Penna,[10] by Martins *et al.*,[11] Rieger *et al.*[12] and Grassberger.[13] Independently, mean-field type approximations of varying complexity were also used to study the DS problem.[12, 14, 15, 16] The original scheme of Martins et al used a single uniformly distributed random number $0 < z < 1$ for the two replicas: using the above definitions this means that the choice

$$r_{01} = r_{10} = \theta(p_1 - z), \qquad r_{11} = \theta(p_2 - z) \tag{6}$$

was made, which can be expressed as

$$\begin{aligned} \tilde{\beta} &= \langle r_{01} r_{10} \rangle = p_1 \\ \tilde{\alpha} &= \langle r_{01} r_{11} \rangle = \text{Min}(p_1, p_2) \end{aligned} \tag{7}$$

The dynamical process is generated by setting

$$\sigma_i(t+1) = r_{\sigma_{i-1}\sigma_{i+1}}, \qquad \sigma'_i(t+1) = r_{\sigma'_{i-1}\sigma'_{i+1}} \tag{8}$$

Kohring and Schreckenberg recognized the fact that one could, in principle, use two different random numbers to determine $\sigma_i(t+1)$ and $\sigma_i(t+1)'$, if at least one of the two neighbor sites was damaged[6] at time t. In fact they studied DS using two different random numbers z_{01} and z_{11}, their DS procedure has two *fully correlated* binary variables (r_{01} and r_{10}) and two *uncorrelated* ones (r_{01} and r_{11}):

$$r_{01} = r_{10} = \theta(p_1 - z_{01}), \qquad r_{11} = \theta(p_2 - z_{11}) \tag{9}$$

the correlations being

$$\begin{aligned} \tilde{\beta} &= \langle r_{01} r_{10} \rangle = p_1 \\ \tilde{\alpha} &= \langle r_{01} r_{11} \rangle = \langle r_{01} \rangle \langle r_{11} \rangle = p_1 p_2 \end{aligned} \tag{10}$$

Tomé[16] was the first to point out that damage could be introduced in different ways to the DK model and that the procedures used in Ref. 9 and in Ref. 14 constituted two particular choices. In fact, the dynamics generated by using on the first replica $\sigma_i(t+1) = r_{\sigma_{i-1}\sigma_{i+1}}$ and $\sigma'_i(t+1) = r_{\sigma'_{i-1}\sigma'_{i+1}}$ on the second gave rise[15] to a shift of the original "phase boundary" (as obtained with a single random number, Eq. (6)). As discussed in Sec. II, the evolution of a single replica is completely insensitive to whether one or two random variables are used in the dynamic procedure, which prompted Grassberger[13] to make his observation quoted in the Introduction.

Finally we note that Grassberger has formulated recently[13] a conjecture, which is a natural extension of previous statements[20, 21, 8] regarding universality of directed percolation transitions for models with non-symmetric absorbing states.[22] According to this conjecture damage-spreading transitions should be in the universality class of directed percolation,[13] provided some general conditions are satisfied. The DK model is a natural candidate to test this conjecture because of its simplicity, ease to simulate and our precise knowledge of the existence of a DS transition and its location. Grassberger presented numerical evidence for his conjecture, which

[6] The possibility of studying damage spreading with entirely different random numbers on the two replicas was raised by Glotzer *et al.*[19] for the Ising model. This notion was never implemented by them and for a good reason. This work is totally irrelevant to ours; we thank a nonanonymous referee for calling our attention to it.

we also confirmed and extended. We also show below that in a region of the p_1, p_2 plane one can map DS exactly to the DK model and hence onto DP. This result is an extension of a statement first made by Kohring and Schreckenberg,[14] who noted that such a mapping holds on the $p_2 = 0$ line.

4. TRUE PHASES IN THE DK MODEL

As discussed in Sec. III, the most general dynamic rule that can be defined for two replicas of the DK automaton, in accordance with these constraints, has two degrees of freedom or parameters, $\tilde{\alpha}$ and $\tilde{\beta}$. As it turns out (see Appendix A), the possible values that $\tilde{\alpha}$ and $\tilde{\beta}$ can take are restricted by requiring that all transition rates have to be positive. For any value of p_1, p_2, the range of allowed values of the parameters $\tilde{\alpha}$ and $\tilde{\beta}$ is given by

$$\begin{aligned} \max(0, p_1 + p_2 - 1) \leqslant \tilde{\alpha} \leqslant \min(p_1, p_2) \\ \max(0, 2\tilde{\alpha} - p_2, 2p_1 - 1, 2p_1 - 1 - 2\tilde{\alpha} + p_2) \leqslant \tilde{\beta} \leqslant p_1 \end{aligned} \tag{11}$$

There are three important special cases, namely those of

- maximal correlations: $\tilde{\alpha} = \min(p_1, p_2)$, $\tilde{\beta} = p_1$
- no correlations: $\tilde{\alpha} = p_1 p_2$, $\tilde{\beta} = p_1^2$
- minimal correlations: $\tilde{\alpha} = p_1 + p_2 - 1$, $\tilde{\beta} = 2p_1 - 1$.

In the case of minimal correlations, the values listed above hold only in the region $2p_1 + p_2 > 2$ (see Appendix A). Note that the when a single random number was used the resulting correlations, Eq. (7), take the maximal possible values.

4.1. Exact Results

We turn now to show that for $p_2/2 \leqslant p_1 \leqslant 1 - p_2/2$ the damage spreading process can be mapped exactly onto a directed percolation process. Kohring and Schreckenberg[14] have shown that such a mapping holds on the *line* $p_2 = 0$. Clearly, their choice of parameters (10) is a particular case of our damage spreading procedure, which is the most general one that satisfies rules 1–3 listed above. Therefore we find a wider (*two-dimensional*) region in the p_1, p_2 plane in which such a mapping is possible. To see this, let $\Delta_i = 1 - \delta_{\sigma_i, \sigma'_i}$ be the damage at site i. By $P_D(\Delta_i = 1 \mid \sigma_{i-1}\sigma_{i+1}; \sigma'_{i-1}\sigma'_{i+1})$ we denote the probability to generate a damaged site for a given initial configuration in a particular update. These

Table I. Probabilities $P_D(\Delta_i = 1 \mid \sigma_{i-1}\sigma_{i+1}; \sigma'_{i-1}\sigma'_{i+1})$ for the Generation of Damage in the DK Model[a]

	$\sigma'_{i-1}, \sigma'_{i+1}$			
$\sigma_{i-1}, \sigma_{i+1}$	00	01	10	11
00	0	p_1	p_1	p_2
01	p_1	0	X	Y
10	p_1	X	0	Y
11	p_2	Y	Y	0

[a] X and Y are defined in Eq. (26).

probabilities are listed in Table I, in which we introduced for brevity the notation

$$X = 2p_1 - 2\tilde{\beta}, \qquad Y = p_1 + p_2 - 2\tilde{\alpha} \tag{12}$$

In general, the probability for generating damage on site i depends on the previous states of both replicas, i.e. on $(\sigma_{i-1}\sigma_{i+1}; \sigma'_{i-1}\sigma'_{i+1})$; knowledge of Δ_{i-1} and Δ_{i+1} does not suffice to determine Δ_i at the next time step. Thus damage spreading itself cannot be seen as an independent process. We may, however, pose the following question: under which conditions will damage spread as if it were generated by an independent process? That is, when do we have

$$P_D(\Delta_i \mid \sigma_{i-1}\sigma_{i+1}; \sigma'_{i-1}\sigma'_{i+1}) = P_D(\Delta_i \mid \Delta_{i-1}, \Delta_{i+1}) \tag{13}$$

In order to satisfy this condition, any two entries in Table I, that correspond to the same initial damage $\{\Delta_{i-1}\Delta_{i+1}\}$, should be equal. For example all four initial configurations

$$\{\sigma_{i-1}\sigma_{i+1}; \sigma'_{i-1}\sigma'_{i+1}\} = (\{11; 00\}, \{10; 01\}, \{01; 10\}, \{00; 11\})$$

have the same initial damage $\{\Delta_{i-1}, \Delta_{i+1}\} = \{1, 1\}$. In order to satisfy Eq. (13), the four entries (p_2, X, X, p_2) must have the same value, i.e. we must have $p_2 = X$. A similar consideration leads to the condition $Y = p_1$; that is, we must have

$$\begin{aligned} &P_D(1 \mid 00) = 0 \\ &P_D(1 \mid 01) = P_D(1 \mid 10) = p_1 = Y \\ &P_D(1 \mid 11) = p_2 = X \end{aligned} \tag{14}$$

Note that these are precisely the update rules of the DK process. Using the definitions (12), we see that the correlations must satisfy

$$\tilde{\alpha} = \frac{p_2}{2}, \qquad \tilde{\beta} = p_1 - \frac{p_2}{2} \tag{15}$$

Since the correlation parameters are restricted by Eq. (11), the allowed range for p_1 and p_2 in which these conditions can hold is a triangle in the phase diagram:

$$p_2/2 \leqslant p_1 \leqslant 1 - p_2/2 \tag{16}$$

To summarize: we have proved that within this triangle we can find correlations $\tilde{\alpha}$, $\tilde{\beta}$ such that the damage spreading process follows the dynamical rules of a single DK automaton. Say we have a line in the (p_1, p_2) plane that lies within this region. For every point (p_1^*, p_2^*) on this line we can find $\tilde{\alpha}$, $\tilde{\beta}$ values for which DS evolves precisely like a DK automaton with parameters (p_1^*, p_2^*). Since part of the transition line of the DK model (from dry to wet phase) lies in the triangle (16), on any trajectory that crosses this part of the phase boundary we will observe a damage spreading transition precisely at the DP transition and with DP exponents (provided we chose $\tilde{\alpha}$, $\tilde{\beta}$ according to Eq. (15).) In particular, this holds for the line $p_2 = 0$, as discovered in Ref. 14; note that for $p_2 = 0$ their choice of correlations, Eq. (10) precisely satisfy Eq. (15).

Outside the triangle (16) it is not possible to find values of $\tilde{\alpha}$, $\tilde{\beta}$ for which this mapping holds exactly. Needless to say this rules out neither the existence of DS transitions, nor their being in the DS universality class.

4.2. Results from Comparing Probabilities

Other useful results can be obtained by comparing probability tables of different pairs of automata. The basic idea is that by increasing (decreasing) *all* probabilities in Table I, damage spreading will be more (less) likely. More precisely, if a pair of DK automata described by parameters p_1^*, p_2^*, $\tilde{\alpha}^*$, $\tilde{\beta}^*$ exhibits damage spreading, we expect that any other pair of automata with parameters p_1, p_2, $\tilde{\alpha}$, $\tilde{\beta}$ satisfying

$$\begin{aligned} p_1^* &\leqslant p_1 \\ p_2^* &\leqslant p_2 \\ p_1^* + p_2^* - 2\tilde{\alpha}^* &\leqslant p_1 + p_2 - 2\tilde{\alpha} \\ 2p_1^* - 2\tilde{\beta}^* &\leqslant 2p_1 - 2\tilde{\beta} \end{aligned} \tag{17}$$

exhibits damage spreading as well. Vice versa, if damage heals in a pair of automata described by p_1^*, p_2^*, $\tilde{\alpha}^*$, $\tilde{\beta}^*$, then for any other pair with p_1, p_2, $\tilde{\alpha}$, $\tilde{\beta}$ obeying

$$\begin{aligned} p_1^* &\geqslant p_1 \\ p_2^* &\geqslant p_2 \\ p_1^* + p_2^* - 2\tilde{\alpha}^* &\geqslant p_1 + p_2 - 2\tilde{\alpha} \\ 2p_1^* - 2\tilde{\beta}^* &\geqslant 2p_1 - 2\tilde{\beta} \end{aligned} \tag{18}$$

we expect damage to heal. Although these statements are very plausible, we were not able to prove them rigorously. However, we performed various numerical tests which turned out to be consistent with the inequalities stated above.

Because of these inequalities the boundaries between the three regions in the phase diagram correspond to extremal correlations $\tilde{\alpha}$ and $\tilde{\beta}$. For example, if at a point (p_1, p_2) damage spreads in a model with maximal correlations $\tilde{\alpha}^{\max} = \min(p_1, p_2)$ and $\tilde{\beta}^{\max} = p_1$, then Eq. (17) implies that damage spreads also for every $\tilde{\alpha}$ and $\tilde{\beta}$ in the allowed range (11). This, however, means that the point (p_1, p_2) belongs to region 1 in the phase diagram. Therefore the phase boundary of region 1 coincides with the DS transition line for maximal correlations. Similarly one can use Eq. (18) to show that the phase boundary between regions 2 and 3 coincides with the DS transition line for minimal correlations. It turns out (see Fig. 2) that

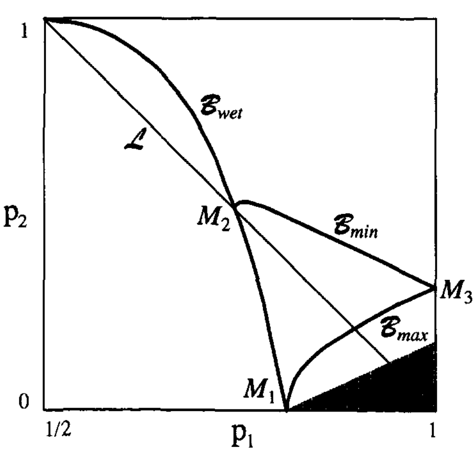

Fig. 2. Schematic phase diagram, displaying the special lines and points that are referred to in the text.

this line lies entirely in the region $2p_1 + p_2 > 2$ so that minimal correlations are well defined (see Appendix B).

Alternatively one can compare the probabilities for generating damage in a pair of DK automata to the probabilities of generating a wet site in a single DK automaton. To this end one simply has to use the same inequalities setting $\tilde{\alpha}^* = p_2^*/2$ and $\tilde{\beta}^* = p_1^* - p_2^*/2$. For example, if p_1^* and p_2^* represent a point in the wet phase of the DK phase diagram, then for all pairs of automata parametrized by p_1, p_2, $\tilde{\alpha}$, $\tilde{\beta}$ and satisfying

$$p_1^* \leqslant p_1, \qquad p_2^* \leqslant p_2, \qquad p_1^* \leqslant p_1 + p_2 - 2\tilde{\alpha}, \qquad p_2^* \leqslant 2p_1 - 2\tilde{\beta} \tag{19}$$

damage will spread. On the other hand, if p_1^* and p_2^* belong to the dry phase of the DK model then in all pairs of automata with

$$p_1^* \geqslant p_1, \qquad p_2^* \geqslant p_2, \qquad p_1^* \geqslant p_1 + p_2 - 2\tilde{\alpha}, \qquad p_2^* \geqslant 2p_1 - 2\tilde{\beta} \tag{20}$$

damage does not spread.

As an illustration of Eq. (19) consider the point M_1 in Fig. 2: Setting $p_1^* = p_1^c \approx 0.809$ and $p_2^* = 0$ we obtain the conditions

$$p_1^c \leqslant p_1, \qquad p_1^c \leqslant p_1 + p_2 - 2\tilde{\alpha}, \qquad 0 \leqslant 2p_1 - 2\tilde{\beta} \tag{21}$$

Using the bounds (11) we find from these inequalities that in the triangle $p_1 - p_2 \geqslant p_1^*$ damage spreads with certainty for any $\tilde{\alpha}$ in the allowed range. In Fig. 2 this triangle is indicated as a shaded region.

4.3. Terminal Points of the Phase Boundaries

We turn now to derive, using the arguments introduced above, a few exact results concerning the phase boundaries for minimal and maximal correlations. As explained above, these boundaries are the transition lines between the damage-spreading phases 1, 2 and 3 described in the Introduction and shown on 1. In order to make our arguments easier to follow, we present in Fig. 2 all the lines and special points that are mentioned.

Let us consider first the case of *maximal correlations*. That is, for every point (p_1, p_2) in the phase diagram we assign $\tilde{\alpha} = \min(p_1, p_2)$ and $\tilde{\beta} = p_1$ and look for the boundary $\mathscr{B}_{\mathrm{max}}$ between the region in which damage spreads and the one in which it doesn't. Denote the wet-to-dry transition line of the DK model by $\mathscr{B}_{\mathrm{wet}}$ (see Fig. 2).

We now prove that $\mathscr{B}_{\mathrm{max}}$ and $\mathscr{B}_{\mathrm{wet}}$ can intersect only at the point $M_1 = (p_1^c, 0)$, where $\mathscr{B}_{\mathrm{wet}}$ intersects the p_1 axis. On this axis maximal correlations correspond to the choice $\tilde{\alpha} = p_2 = 0$ and $\tilde{\beta} = p_1$, which also satisfy the conditions (15), i.e. the damage spreading process can be mapped onto a single DK model. Therefore we know that a DS transition will occur precisely at $p_1 = p_1^c$, so that at all points $(p_1 > p_1^c, p_2 = 0)$ we must have DS. On the other hand, as we will now show, for maximal correlations there cannot be DS on any point on $\mathscr{B}_{\rm wet}$; hence the boundary $\mathscr{B}_{\rm max}$ must go through M_1. In order to substantiate the last claim, to see this, note that (for the region of interest, $p_1 > 1/2$) maximal correlations imply the inequalities

$$p_1 + p_2 - 2\tilde{\alpha} = |p_1 - p_2| < p_1, \qquad 2p_1 - 2\tilde{\beta} = 0 < p_2 \tag{22}$$

For a point (p_1, p_2) on $\mathscr{B}_{\rm wet}$ these inequalities, when used together with Eq. (20), imply that damage does not spread. Furthermore, since (22) are (for $p_2 > 0$) strict inequalities, $\mathscr{B}_{\rm wet}$ lies *inside* the no-spread region. The point M_1 is on the boundary between this region and a region (containing the $p_1 > p_1^c$ axis) in which damage does spread; therefore M_1 must lie on $\mathscr{B}_{\rm max}$.

We turn now to the phase boundary $\mathscr{B}_{\rm min}$ for *minimal correlations*, and show that it terminates at the point M_2, where $\mathscr{B}_{\rm wet}$ intersects the line $\mathscr{L}$, given by $p_1 + p_2/2 = 1$. First notice that on $\mathscr{L}$ minimal correlations correspond to $\tilde{\alpha} = 1 - p_1$ and $\tilde{\beta} = 2p_1 - 1$. Therefore (see Eq. (15)) the DS process is equivalent, on $\mathscr{L}$, to a DK model, so that as we move on $\mathscr{L}$, keeping minimal correlations, a DS transition occurs at the point $M_2 = (p_1^*, p_2^*)$, where $\mathscr{L}$ intersects $\mathscr{B}_{\rm WET}$. As before, we show next that at all points on $\mathscr{B}_{\rm WET}$ with $p_2 > p_2^*$ damage doesn't spread; hence $\mathscr{B}_{\rm min}$ must pass through M_2. To prove the last claim note that the segment of $\mathscr{B}_{\rm wet}$ that lies above the intersection point is in the region where $p_1 + p_2/2 > 1$ and this immediately leads (for minimal correlations) to the inequalities

$$p_1 + p_2 - 2\tilde{\alpha} = 2 - p_1 - p_2 < p_1, \qquad 2p_1 - 2\tilde{\beta} = 2 - 2p_1 < p_2 \tag{23}$$

According to Eqs. (20), this implies that on the DK transition line in the region $p_1 + p_2/2 > 1$ damage does not spread. Therefore M_2 is the terminal point of $\mathscr{B}_{\rm min}$.

Having located the endpoints M_1 and M_2, we now turn to the opposite end of the lines $\mathscr{B}_{\rm max}$ and $\mathscr{B}_{\rm min}$. Note that for $p_1 = 1$ the bounds (11) collapse to $\tilde{\alpha} = p_2$, $\tilde{\beta} = 1$, i.e. maximal and minimal correlations are identical and hence $\mathscr{B}_{\rm max}$ and $\mathscr{B}_{\rm min}$ meet at some point M_3 on the $p_1 = 1$ line. The three special "multicritical" points discussed above determine the topology of the phase diagram for DS. In order to obtain high precision quantitative information about the location of the transition lines we performed numerical studies of damage spreading in the DK model.

4.4. Numerical Results

In order to obtain accurate numerical estimates for the critical parameters of models with absorbing states one usually has to let the system evolve for extremely long times.[9] Grassberger overcame the difficulty posed by long transients and obtained good statistics by simulating n replicas of the same system in parallel, using simple bit manipulations on computers with unsigned words of length n.[13] Using this multi-spin coding method he measured the decay in damage on a one-million site chain, allowing it to evolve for hundreds of thousands of time steps. Because of the improved statistics he was able to determine the critical exponents for damage spreading at a particular transition point with high accuracy.

Another method to determine the critical point efficiently is the so-called gradient method which was introduced by Zebende and Penna.[10] In this method a gradient in p_1 and p_2 is arranged along the chain. The values of the parameters at the two end-points of the chain are chosen to be in different phases, i.e. on different sides of the transition point. This allows the critical point to be determined by measuring the average location of the boundary of the active (damaged) cluster.

In the present work we used a combination of multi-spin coding and the gradient method. In combining these methods, a number of problems emerged which we solved as follows:

1. In order to measure the damage spreading transition point, one has to find the first position (approaching from the non-spreading phase) where damage occurs. Simulating $n = 64$ lattices in parallel, this has to be done for each of the $64 \cdot 63/2 = 2016$ pairs of replicas. To do this one has to set up a 64×64 table in order to keep track of damaged pairs. Moreover, one has to scan the words bit by bit which makes it impossible to use parallel bit manipulations. The large amount of CPU time needed for this process usually kills the advantage one gains from the multi-spin encoding. In order to solve this problem, we used a simplified search algorithm which is based on fast bit operations. The price we pay is that only $\approx 75\%$ of all possible pairs are taken into account.[7] We proved that the error of this method does not bias the measurement of the transition point.

[7] In this approximation, a given replica is declared to be damaged at site i if the majority of the other replicas is in a different state. Instead of keeping track of damaged *pairs of replicas*, we recognize only *single replicas* from where damage originated. This amounts in dropping statistically 25% of all possible pairs.

2. Zebende and Penna started each run with a single damaged seed located somewhere on the chain. It is not clear whether the choice of the location influences the results. In order to circumvent this problem, we used initial conditions with randomly distributed damage all over the chain. This amounts to half the sites of a pair of chains being damaged initially. The results did not depend on the amount of initial damage.
3. The gradient method is a finite-size simulation and therefore boundary conditions may play an important role. In the work of Zebende and Penna the boundary conditions can be understood as dry walls and it is not clear to what extent they affect the measurements. In order to minimize this effect, we created, on a chain of $2N$ sites, a gradient with reflection symmetry, $[p_1(i) = p_1(1) + (i-1)\,\delta p_1$ and $p_1(2N-i+1) = p_1(i)$ for $i = 1, 2,..., N]$, keeping $p_2 = \text{const.}$, and measured the boundary of the damaged cluster on both sides. Alternatively, the roles of p_1 and p_2 were reversed. We expect finite-size effects to be less important for these periodic boundary conditions.

The phase diagram of the DK automaton, obtained using the *multiple lattice gradient method*, is presented in Fig. 1. First, we verified numerically the prediction that larger correlations correspond to smaller damage and vice versa. This was done by scanning the $(\tilde{\alpha}, \tilde{\beta})$ space for various points in the (p_1, p_2) plane. Next we determined the DS transition lines for minimal and maximal correlations. Typical gradient values of $1.2 \cdot 10^{-5}$ were used for lattice sizes $L = 8192$ and upwards. A transient period of at least $2L$ was followed by an averaging period of L time steps. For (p_1, p_2) near the transition lines longer transient times were used. The terminal points of the phase boundaries were determined with high accuracy. Using a chain with $L = 16384$ sites, gradients down to $1.22 \cdot 10^{-6}$ and transients of 231072, we measured the following critical values at these special points: $p_1^c = 0.8087(5)$ (on the $p_2 = 0$ line); $p_2^c = 0.3130(5)$ (on the $p_1 = 0$ line). The new triple point was located at $p_1^* = 0.744(10)$, $p_2^* = 0.526(10)$. This was done *without* using our analytic result that identified this point as the crossing of $\mathcal{L}$ with $\mathcal{B}_{\text{wet}}$; the value of $p_1^* + p_2^*/2 = 1.007(11)$ agrees with the predicted value (of 1) for points on $\mathcal{L}$.

Measuring the density of damage along the gradient of the chain, we could estimate the density exponent β. At the terminal points we found $\beta = 0.302(30)$ for $p_2 = 0$ and $\beta = 0.296(30)$ on the $p_1 = 1$ line. We also measured the exponent at a point $(p_1, p_2) = (0.85, 0.35)$ which lies inside phase 3. This was done by crossing the DS phase boundary while varying the correlations $\tilde{\alpha}$ and $\tilde{\beta}$, yielding the value $\beta = 0.279(10)$. All results are in fair agreement with the expected density exponent of directed percolation $\beta = 0.277(1)$.[23, 24]

5. SUMMARY

We have rules that a most general damage spreading procedure should satisfy. These rules are most natural: they ensure that the evolution of a single replica is not affected by the fact that two replicas are evolving simultaneously; that the range of damage spreading does not exceed the range of interactions in the original single model and that the two evolving replicas respect the symmetries of the model. These rules can be cast in a formal setting, that enables us to study damage spreading in terms of correlation coefficients between various stochastic binary variables. Thus we are considering *all possible damage spreading procedures* and identify different damage spreading *phases* in terms of the manner in which this complete set of procedures behaves. Three possible phases can occur; one in which damage spreads for *all* allowed procedures, one in which it does not spread for any procedure and the third, in which for some procedures damage spreads while for others it does not.

These ideas were implemented for the Domany–Kinzel automaton, for which the three phases were identified, using a combination of numerical and analytic methods. We have shown that in an extended region of the model's parameter space damage spreading can be mapped onto the evolution of the DK automaton itself. This observation supports Grassberger's recent conjecture to the effect that damage spreading is in the directed percolation universality class. This was also confirmed by numerical tests (performed in regions where the above mentioned mapping does not hold).

APPENDIX A. GENERATION OF CORRELATED RANDOM VARIABLES

In this Appendix we explain in detail how correlated random variables r_{01}, r_{10} and r_{11}, that govern the evolution of the DK-model, can be generated. We also prove the allowed ranges for $\tilde{\alpha}$ and $\tilde{\beta}$, given in Eq. (11). Finally, we explain the manner in which minimal correlations are given by the expression presented in Sec. V.

Since in each update r_{01}, r_{10}, and r_{11} can be either zero or one, there are eight possible combinations. By $\pi_{r_{01}, r_{10}, r_{11}}$ we denote the (positive) probability to generate the combination $\{r_{01}, r_{10}, r_{11}\}$. These probabilities are normalized such that they sum up to one. The random variables r_{01}, r_{10}, and r_{11} can be generated by taking one uniformly distributed random number $0 < z < 1$ and selecting one of the eight possible outcomes according to the probabilities $\pi_{r_{01}, r_{10}, r_{11}}$. The correlation functions between the random variables can be represented in terms of the π's, for example $\pi_{111} = \langle r_{01} r_{10} r_{11} \rangle$ and $\pi_{110} + \pi_{111} = \langle r_{01} r_{10} \rangle$. Collecting all identities of this type, we obtain seven equations:

$$\begin{aligned} \pi_{100} + \pi_{101} + \pi_{110} + \pi_{111} &= p_1 \\ \pi_{010} + \pi_{011} + \pi_{110} + \pi_{111} &= p_1 \\ \pi_{001} + \pi_{011} + \pi_{101} + \pi_{111} &= p_2 \\ \pi_{011} + \pi_{111} &= \tilde{\alpha} \\ \pi_{101} + \pi_{111} &= \tilde{\alpha} \\ \pi_{110} + \pi_{111} &= \tilde{\beta} \\ \pi_{111} &= \tilde{\gamma} \end{aligned} \tag{24}$$

Together with the normalization these equations determine all probabilities $\pi_{r_{01}, r_{10}, r_{11}}$:

$$\begin{aligned} \pi_{000} &= 1 - \tilde{\gamma} + \tilde{\beta} + 2\tilde{\alpha} - 2p_1 - p_2 \\ \pi_{001} &= p_2 - 2\tilde{\alpha} + \tilde{\gamma} \\ \pi_{010} &= \pi_{100} = p_1 - \tilde{\alpha} - \tilde{\beta} + \tilde{\gamma} \\ \pi_{011} &= \pi_{101} = \tilde{\alpha} - \tilde{\gamma} \\ \pi_{110} &= \tilde{\beta} - \tilde{\gamma} \\ \pi_{111} &= \tilde{\gamma} \end{aligned} \tag{25}$$

Since all π have to be positive, we obtain six inequalities:

$$\max(0, 2\tilde{\alpha} - p_2) \leqslant \tilde{\gamma} \leqslant \tilde{\alpha} \tag{26}$$

$$\max(\tilde{\gamma}, \tilde{\gamma} - 1 - 2\tilde{\alpha} + 2p_1 + p_2) \leqslant \tilde{\beta} \leqslant \tilde{\gamma} + p_1 - \tilde{\alpha} \tag{27}$$

For a given choice of the parameters p_1, p_2 these inequalities imply restrictions on the correlation parameters $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$. The allowed range of these parameters can be derived as follows. First let us consider the restrictions on $\tilde{\alpha}$. Eq. (26) implies that $0 \leqslant \tilde{\alpha} \leqslant p_2$ whereas Eq. (27) leads to the condition $p_1 + p_2 - 1 \leqslant \tilde{\alpha} \leqslant p_1$. Both of them can be combined by requiring

$$\max(0, p_1 + p_2 - 1) \leqslant \tilde{\alpha} \leqslant \min(p_1, p_2) \tag{28}$$

For a given $\tilde{\alpha}$ in this interval the maximal ranges of $\tilde{\beta}$ and $\tilde{\gamma}$ are given in Eqs. (26)–(27). However, since we do not explicitly use the three-point correlation parameter $\tilde{\gamma}$, we are only interested in the maximal range of $\tilde{\beta}$. This range can be obtained by inserting the extremal values for $\tilde{\gamma}$ into Eq. (27), that is $\tilde{\gamma} = \max(0, 2\tilde{\alpha} - p_2)$ on the l.h.s. and $\tilde{\gamma} = \tilde{\alpha}$ on the r.h.s. Thus for a given $\tilde{\alpha}$ in the range (28) the corresponding maximal range of $\tilde{\beta}$ is:

$$\max(0, 2\tilde{\alpha} - p_2, 2p_1 - 1, 2p_1 + p_2 - 1 - 2\tilde{\alpha}) \leqslant \tilde{\beta} \leqslant p_1 \tag{29}$$

In other words, if $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy Eqs. (28)–(29), we are able to find some $\tilde{\gamma}$ such that all probabilities $\pi_{r_{01}, r_{10}, r_{11}}$ are positive.

APPENDIX B. MAXIMAL AND MINIMAL CORRELATIONS

In order to determine the phase boundaries $\mathscr{B}_{\max}$ and $\mathscr{B}_{\min}$, one has to identify situations where the correlations are extremal. For given parameters p_1, p_2 maximal correlations (minimal damage) simply correspond to taking the upper bounds of the intervals (28) and (29):

$$\tilde{\alpha} = \min(p_1, p_2), \qquad \tilde{\beta} = p_1 \tag{30}$$

In case of minimal correlations (maximal damage) the situation is more complicated since the correlation parameter $\tilde{\alpha}$ appears on the l.h.s. of Eq. (29) with a negative sign. Therefore by increasing $\tilde{\alpha}$, the minimal value of $\tilde{\beta}$ may decrease which makes it impossible to predict where damage is maximal. However, we can show that in the triangular region *above* line $\mathscr{L}$ in Fig. 2, where $2p_1 + p_2 - 2 \geqslant 0$, this problem does not arise. In this region Eq. (28) implies that $\tilde{\alpha} \geqslant p_1 + p_2 - 1$ and therewith $2\tilde{\alpha} \geqslant 2p_1 + 2p_2 - 2 \geqslant p_2$. Hence according to Eq. (26) the minimal value that $\tilde{\gamma}$ can have is $2\tilde{\alpha} - p_2$. Inserting this value into Eq. (27) we get the inequality

$$\max(2\tilde{\alpha} - p_2, 2p_1 - 1) \leqslant \tilde{\beta} \leqslant p_1 \tag{31}$$

which replaces Eq. (29) in the specified triangle. In this inequality $\tilde{\alpha}$ occurs with a positive sign on the l.h.s. and therefore the case of minimal correlations is well defined:

$$\tilde{\alpha} = p_1 + p_2 - 1, \qquad \tilde{\beta} = 2p_1 - 1 \tag{32}$$

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